Tutorial 4

February 16,2017

1. Discuss why there is no maximum principle for wave equation?

   Consider the following Cauchy problem:

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0, & -\infty < x < +\infty, \quad t > 0 \\ u(x, t = 0) = 0, & \partial_t u(x, t = 0) = \sin x, \quad -\infty < x < +\infty \end{cases}$$

   And the unique solution is given by d'Alembert formula:

$$u(x,t) = \frac{1}{2}\cos(x+t) - \cos(x-t) = -\sin x \sin t, \quad -\infty < x < \infty, t > 0$$

   Then $u(x,t)$ attains its maximum 1 only at the interior points $(\frac{\pi}{2} \pm 2n\pi, \frac{3\pi}{2} + 2n\pi)$ or $(\frac{3\pi}{2} \pm 2n\pi, \frac{\pi}{2} + 2n\pi)$ for $n = 0, 1, 2, \cdots$. However, $u(x,t) = 0$ on the boundary $\{(x,t) : t = 0\}$. Therefore there is no maximum principle for the Cauchy problem for the 1-dimensitonal wave equation.

   Remark: The key is to find an counterexample.

2. Use the Green's function of the heat equation to show that the backward heat equation is not well-posed.

   Note that $S(x,t)$ satisfies $u_t = ku_{xx}$ for any $t > 0$, and $S(0,t) \to \infty$ as $t \to 0^+$. Then $u(x,t) = S(x, t+1)$ solves $u_t = ku_{xx}$ for $t > -1$. Then $S(0,t) \to \infty$ as $t \to -1^+$, which implies that there is no solution for the backward heat equation with initial data $u(x,0) = S(x,1) = \frac{1}{\sqrt{4k\pi}} e^{-\frac{x}{4k}}$, hence the backward heat equation is not well-posed.

3. Let $\phi(x)$ be a continuous function such that $|\phi(x)| \le Ce^{ax^2}$. Show that formula (8) on page 48 for the solution of the diffusion equation makes sense for $0 < t < \frac{1}{4ak}$, but not necessarily for larger t.

   Solution: Since

$$|e^{-(x-y)^2/4kt}\phi(y)| \le Ce^{-(x-y)^2/4kt + ay^2} = Ce^{(a - \frac{1}{4kt})y^2 + \frac{x}{2kt}y - \frac{x^2}{4kt}},$$

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y)\, dy$$

   makes sense for $a - \frac{1}{4kt} < 0$, i.e. $0 < t < 1/(4ak)$, but not necessarily for large t, for example, $\phi(x) = e^{ax^2}$.

4. Use energy method to show that the energy for diffution eqution decays with a rate for large time.

   Multiplying $\partial_t v = k\partial_x^2 v$ by v and then integrating w.r.t x give that

$$\frac{d}{dt}\int_{-\infty}^{\infty} \frac{1}{2}|v|^2 dx = \int_{-\infty}^{\infty} k\partial_x^2 v v dx$$

   It follows from integration by parts that

$$\frac{d}{dt}\int_{-\infty}^{\infty} \frac{1}{2}|v|^2 dx = k\partial_x v v\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} k(\partial_x v)^2 dx = -\int_{-\infty}^{\infty} k(\partial_x v)^2 dx \le 0$$

   for any $t \ge 0$. Here assume that v vanishes when $x \to \infty$. Hence if the solution is not a constant, $\frac{d}{dt}\int_{-\infty}^{\infty} \frac{1}{2}|v|^2 dx < 0$, then the energy $E = \int_{-\infty}^{\infty} \frac{1}{2}|v|^2 dx$ decays as $t \to \infty$.